

On Newtonian frames

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Abstract. In Newtonian space-time there exist four, and only four, causal classes of frames. Natural frames allow to extend this result to coordinate systems, so that coordinate systems may be also locally classified in four causal classes. These causal classes admit simple geometric descriptions and physical interpretations. For example, one can generate representatives of the four causal classes by means of the *linear synchronization group*. Of particular interest is the *local Solar time synchronization*, which reveals the limits of the frequent use of the concept of ‘causally oriented coordinate’, such as that of ‘time-like coordinate’. Classical *positioning systems*, based in sound or light signals, are, by themselves, interesting examples of location systems, i.e. of physically constructible coordinate systems. They show that one can locate events in Newtonian space-time *without* any use of the concept of synchronization. In fact, the coordinate systems associated to positioning systems, belong to all the classes but the standard one, i.e. the one based in the simultaneity synchronization. The relativistic analogs of these examples, emphasize the contrast between the four Newtonian and the one hundred and ninety nine Lorentzian causal classes of frames of classical and relativistic space-times, respectively.

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1. Introduction

Location systems are physical realizations of coordinate systems. From laboratory domains, Earth surface physics or global navigation systems to space physics, solar system or celestial astronomy, location systems allow the explicit construction of the correspondence between the events of the observable physical world and the points of its mathematical space-time model in the physical theory in use.

A location system must include the protocols for the physical construction of the coordinate lines, coordinate surfaces or coordinate hypersurfaces of the coordinate system that it physically realizes. Thus, for example, these coordinate elements may be realized, among other ways, by means of clocks for timelike lines, laser pulses for

null lines, synchronized inextensible threads for spacelike lines, inextensible threads or laser beams for time like surfaces, light-front signals for null hypersurfaces and so on. The point of interest here is that every protocol physically realizes coordinate lines, coordinate surfaces or coordinate hypersurfaces of specific *causal orientations*. Conversely, the causal orientation of the ingredients of a coordinate system intimately constraints the physical protocols needed for the construction of the corresponding location system.

The different protocols involved in the construction of location systems give rise to coordinate elements (lines, surfaces and hypersurfaces) of different causal orientations, i.e. they realize coordinate systems of different causal nature. It is known that the number of coordinate systems of different causal nature that can be constructed in relativistic space-times is of exactly one hundred and ninety nine [1]. But the corresponding question for the Newtonian space-time has never been asked until recently [2].

Here this question is analyzed and it is shown that, in strong contrast with the relativistic case, the number of Newtonian coordinate systems of different causal nature reduces drastically to four.

A precise geometric description of these four classes is given and some possible physical realizations of every one of them are commented. Also, some examples are constructed of coordinate systems for every one of these causal classes. And finally the four causal classes of Newtonian coordinate systems are contrasted with the one hundred and ninety nine Lorentzian causal classes and, among them, specifically with their four relativistic analogs.

1.1. Interest and applications of the causal classification of frames

The interest of the causal classification of coordinate systems is not only taxonomic.

So, for example, in a similar way as three-dimensional Cartesian coordinates frequently induce or are induced by a floor plan and elevation cut of the space, every four-dimensional coordinate system may be seen as a specific cut or foliation of (a region of) the space-time in particular pieces: those defined by the coordinate hypersurfaces, surfaces or lines of the coordinate system. But now these cuts or foliations may be of different specific causal classes. In this sense, the well known usual coordinate systems, essentially based in a three-space foliation plus a one-time congruence, are induced by, or induce, the standard *evolution conception* of Newtonian and relativistic physics. But other cuts or foliations, among the other three possible cuts or foliations in Newtonian theory or among the other one hundred and ninety eight possible cuts or foliations in relativity, may help us to better describe and understand other aspects of the space-time, and even to wake up our interest for variations of physical fields other than the timelike ones, intimately induced by the evolution conception.

But perhaps the most imminent interest of the causal classification of coordinate systems is appearing in the at present methods for solving practical relativistic problems.

Relativity theory is conceptually considered as a physically autonomous theory, i.e. a theory that, for its development, needs no other physical concepts than the ones contained in its specific foundations, or those that can be coherently deduced from them. But in practice, in spite of the efforts made in this direction [2, 3, 4, 5, 6, 7, 8, 9, 10], the development of the least physical practical application needs, for the moment, a detour to Newtonian concepts and post-Newtonian methods. This situation reduces relativity theory, up to little exceptions, to the role of a *corrective algorithm* for Newtonian theory, relegating its best specific concepts to a simple historically astute, but otherwise ineffective, method of setting the main equations of the theory, the Einstein equations. In fact, irrespective of the revolutionary and paradigmatic concepts that general relativity opposed to the Newtonian scope of the space-time, only quantitative first terms in Taylor development of Einstein equations with respect to a Newtonian background remain essentially the unique element of general relativity used to improve Newtonian results obtained under Newtonian concepts.

As long as this situation remains, it is highly convenient in post-Newtonian developments to choose location or coordinate systems such that their causal properties be the same both for the relativistically corrected metric structure as well as for the starting Newtonian one. Otherwise, in going from Newtonian to relativistic results by the addition of higher corrective terms, one would add, to the quantitative corrective process involving the physical quantities of the problem, qualitative corrections due to an eventual change of causal orientations of the coordinate elements of the location system. If such a change takes place, the physical interpretation of the vector or tensor *components* of the physical quantities of the problem, and therefore the adequate instruments for their measure, could change drastically[‡].

Fortunately this convenient choice of analogous causal classes has been made up to now, naturally but unconsciously. Simply because the starting Newtonian coordinate system has been essentially chosen to be the Cartesian one, and that the weak gravitational fields usually considered in astronomy have been unable to change, with the lower order perturbed relativistic values of the metric, their causal orientation. But new problems, concerning black holes, binary systems, gravitational waves, positioning systems, formation flight satellites and space physics, could induce to start from other Newtonian coordinate systems, best adapted to these problems or to push away higher order terms. And then, changes in the causal orientation of some of the ingredients of the starting Newtonian coordinate system become possible when evaluated with the corrective algorithm generating the relativistic space-time metric.

In fact, in numerical relativity, a verification not only of the regularity but of the

[‡] Think that, for example, of the four-dimensional energy tensor, the usual interpretation of their components in terms of energy density, momentum density and stress quantities is only valid for *standard* frames. Standard frames privilege *one* observer among all others, but constitute a little class among the one hundred and ninety nine classes of possible frames; in all the others, and in particular in the real null frames of emission coordinates (see below in the text), such an interpretation fails, because no observers are necessary at all.

stability (constancy) of the whole causal class of the coordinate system would be also convenient in order to guarantee the physical interpretation, at least, of the components of the energetic quantities present in Einstein equations.

These are the main points of interest involving related causal classes of Newtonian *and* relativistic coordinate systems. Other points of interest concerning specifically relativistic coordinate systems were mentioned in [1].

But, in order to better understand the role that location systems as physical objects, or coordinate systems as mathematical objects, play in the conception and analysis of experimental situations, a lot of work remains to be done, the present one being only one of the first little pieces. Recently considered *emission coordinates* go in this direction (see [8, 9, 10] and references therein).

1.2. Structure of the present work

The paper is organized as follows. In Sec. 2 the notion of causal class of a frame is introduced and extended to coordinate systems. Sec 3 characterizes the four causal classes of frames or coordinate systems in Newtonian space-time, and extends this result to arbitrary dimension. In Sec. 4 the notions of coordinate parameter and gradient coordinate are emphasized in order to better understand the limits of the assignation of a causal character to the coordinates, and the first elements of the synchronization group are stressed for the incoming applications. Sec. 5 presents some physical examples of Newtonian coordinates of the four causal classes. It is shown that the linear synchronization group is able to generate coordinate systems of *any* of the four causal classes, the causal class of the ancestral local Solar time is obtained and commented, and Newtonian emission coordinates generated by positioning systems, able to locating events out of any notion of synchronization, are shown to belong to any causal class but the usual one. In Sec. 6 Newtonian and Lorentzian classes are contrasted across the relativistic analogs of the chosen Newtonian examples. Finally, in Sec. 7 we comment on the role that our results can play as training toys for a better understanding of the physical space-time.

Some preliminary results about this work were presented as a contributing lecture at the school on *Relativistic Coordinates, Reference and Positioning Systems* [2].

2. Notion of causal class

In relativity, directions and planes or hyperplanes of directions at an event are said to be spacelike, null or timelike *oriented* if they are respectively exterior, tangent or secant to the light-cone of this event. These *causal orientations*, of clear geometrical and physical meaning, extend naturally to vectors and volume forms on these sets of directions.

Thus, every one of the vectors v_A of a frame $\{v_A\}$ ($A = 1, \dots, 4$) has a particular causal orientation c_A . What about the causal orientations C_{AB} ($A < B$) of the six *associated* planes $\Pi(v_A, v_B)$ of the frame? Are they determined by the sole causal

orientations c_A of the vectors of the frame? Certainly not, because for example the plane associated to two spacelike vectors may have any causal orientation. So, in general, the specifications c_A and C_{AB} are independent.

Moreover, in order to give a complete description of the causal properties of the frames, one needs also to specify the causal orientations c_A of the four covectors θ^A giving the dual frame $\{\theta^A\}$, $\theta^A(v_B) = \delta_B^A$. The c_A 's are one-to-one related to the causal orientations of the four associated 3-planes $\Pi(v_B, v_C, v_D)$ with $\theta^A(v_B) = \theta^A(v_C) = \theta^A(v_D) = 0$ which are not determined, in general, by the specification of both c_A and C_{AB} .

The set of $(4 + 6 + 4 =) 14$ causal orientations $\{c_A, C_{AB}, c_A\}$ is called the *causal signature* of a frame $\{v_A\}$, and characterizes completely its *causal class*: the causal class of a frame is the set of all the frames that have same causal signature. The causal signature of a frame provides exhaustive information about the causal properties of its geometric elements (directions, planes and hyperplanes). Elsewhere [1], the following result was obtained.

Theorem 1 *In a four-dimensional Lorentzian space-time there exist 199 causal classes of frames.*

As a *natural frame* is nothing but the set of derivations along the parameterized lines of a coordinate system, the notion of causal class extends naturally to the set of coordinate lines of the coordinate system and so, to the coordinate system itself. But because this extension of the notion of causal class to a coordinate system is by construction a point by point extension, i.e. the causal class of a coordinate system is the causal class of its natural frame at every point, a coordinate system may present different causal classes at different points of its domain of definition. Indeed, some examples of this situation will be given below.

The assignment of one specific causal class to a coordinate system in a region of the space-time supposes that the causal orientations of all the geometric elements of the coordinate system (lines, surfaces and hypersurfaces) are the same at any point of the region or, in other words, that the region under consideration is a *causal homogeneous region* for the coordinate system in question.

Theorem 1 equivalently states that there are 199 causally different ways to parameterize the events of a relativistic space-time causal homogeneous region. The complete and explicit specification of them was given in [1] and more recently in [2].

By definition, the causal class of a coordinate system $\{x^\alpha\}_{\alpha=1}^4$ in a domain is the causal class $\{c_\alpha, C_{\alpha\beta}, c_\alpha\}$ of its associated natural frame at the events of the domain. The c_α 's are the causal orientations of the vectors $\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha}$ of the natural frame $\{\partial_\alpha\}$ itself, and the c_α 's are the causal orientations of the 1-forms dx^α of the coframe $\{dx^\alpha\}$. Four families of coordinate 3-surfaces (hypersurfaces) are associated with this coframe, and their mutual intersections give six families of coordinate 2-surfaces (surfaces) whose causal orientations are precisely given by $C_{\alpha\beta}$ (of course, the mutual intersections of these surfaces give the four congruences of coordinate lines of causal orientation

c_α). We have chosen the following order for the causal orientations of a causal class: $\{c_1 c_2 c_3 c_4, C_{12} C_{13} C_{14} C_{23} C_{24} C_{34}, c_1 c_2 c_3 c_4\}$.

What is the situation in Newtonian physics concerning causal orientations and causal classes? Of course, now the causal orientations c_A , C_{AB} , c_A reduce to be only of timelike or spacelike character. But a causal class needs also to be characterized by the fourteen quantities $\{c_A, C_{AB}, c_A\}$. Nevertheless now some of them determine systematically the others. Specifically, we shall show in Section 3 that for Newtonian frames one has the implications

$$\{c_A\} \Rightarrow \{C_{AB}, c_A\} \quad , \quad \{C_{AB}\} \Rightarrow \{c_A\} \quad ,$$

but

$$\{C_{AB}\} \nRightarrow \{c_A\} \quad , \quad \{c_A\} \nRightarrow \{c_A, C_{AB}\} \quad .$$

These implications lead to a Newtonian situation remarkably simpler than the Lorentzian one. In fact, surprisingly enough at first glance, only four causally different classes of frames or coordinate systems are admissible in Newtonian space-time (see Sec. 3 below). It is startling that, in spite of this poverty of classes, only the *standard class* (i. e. the one wholly adapted to the absolute space \oplus time Newtonian decomposition) has been explicitly referred to in the literature. In the next section we construct these four classes of Newtonian frames.

3. Causal classes of Newtonian frames

The differences in the geometric description of Lorentzian and Newtonian frames come from the causal structure induced by the metric description of the underlying physics.

In Relativity the space-time metric defines a one-to-one correspondence between vectors and covectors at every event. In contrast, in Newtonian physics no non-degenerate metric structure exists. The degenerate metric structure is given by a rank one covariant positive *time metric* T and an orthogonal rank three contravariant positive *space metric* γ^* , $T \times \gamma^* = 0$, where \times stands for the cross product[§].

The time metric T is necessarily of the form $T = \theta \otimes \theta$, where the 1-form θ , the *time current*, defines the unit of time. That this time is *uniform* for any observer, or *absolute*^{||}, implies the exact character of the time current, $\theta = dt$, where t is any absolute time scale[¶]. The hypersurfaces $t = \text{constant}$ constitute the *instantaneous spaces, simultaneity loci* or *spaces* at the instant t .

It should be stressed that the above elements, T (or θ) and γ^* , already determine the Newtonian causal structure⁺ Here, we are interested only in the causal orientation

§ The cross product \times , or matrix product, is the contraction of the adjacent vector spaces of the tensor product \otimes . In tensor components, $T \times \gamma^*$ is written as $T_{\alpha\rho} \gamma^{*\rho\beta}$.

|| Absolute and uniform times are strongly related. See [11].

¶ A time scale is a rhythm generated by a unit interval together with a choice of origin.

⁺ Nevertheless, for the formulation of the equations of motion, a flat and symmetric affine connection is also required in order to introduce inertia. In addition, in the four-dimensional formulation of

at every event of directions, planes and hyperplanes induced by the sole Newtonian structure provided by θ and γ^* . In this structure, a vector v is *spacelike* if it is instantaneous with respect to the time current θ , i.e. if $\theta(v) = 0$. Otherwise, the vector is *timelike*. A timelike vector v is *future* (resp. *past*) *oriented* if $\theta(v) > 0$ (resp. $\theta(v) < 0$). Obviously, these notions apply naturally to vector fields in causal homogeneous regions.

It is clear that a basis can have *at most* three spacelike vectors so that, denoting with Roman letters (e, t) the causal orientations (respectively spacelike, timelike) of vectors, it holds:

Lemma 1 *Attending to the causal orientation of their vectors, there exist four causal types of Newtonian bases, namely: {teee}, {ttee}, {ttte}, {tttt}.*

In a Newtonian structure, correspondingly, a covector $\omega \neq 0$ is *timelike* if it has no instantaneous part with respect to the space metric γ^* , i.e. if $\gamma^*(\omega) = 0$. Otherwise, the covector ω is *spacelike*. The sole timelike codirection is that defined by the current θ at every event because γ^* has rank 3. Thus, if ω is timelike it is necessarily of the form $\omega = a\theta$ with $a \neq 0$. Then ω is *future* (resp. *past*) *oriented* if $a > 0$ (resp. $a < 0$). Obviously, these notions are also naturally valid for 1-forms in causal homogeneous regions.

It is then clear that a cobasis has *at most* one timelike covector so that, denoting with Italic letters (*e*, *t*) the causal orientations (respectively spacelike, timelike) of covectors, it holds:

Lemma 2 *Attending to the causal orientation of their covectors, there exist two causal types of Newtonian cobases, namely: {teee}, {eeee}.*

Lemmas 1 and 2 show the lack of symmetry of causal types of Newtonian bases and cobases, in contrast to the rigorous symmetry of the relativistic case.

A r -plane Π is *spacelike* if every vector v in it is spacelike. Otherwise, Π is *timelike*, i.e. it contains timelike vectors. Two (resp. three) linearly independent spacelike vectors generate a spacelike 2-plane (resp. 3-plane).

A r -coplane Ω is *timelike* if it contains the time current θ . Otherwise Ω is *spacelike*.

The annihilator coplane Ω_Π of a r -plane Π is the $(4 - r)$ -coplane

$$\Omega_\Pi \equiv \{\omega \mid \omega(v) = 0 \ \forall v \in \Pi\}.$$

Obviously, these definitions apply also to r -plane fields and r -coplane fields in causal homogeneous regions.

Accordingly, we have the following result.

Lemma 3 *A r -plane Π is spacelike (resp. timelike) iff Ω_Π is timelike (resp. spacelike).*

Newtonian gravity, the requirement of another symmetric, non-flat and not metric connection is needed in order to introduce the gravitational field [12, 11, 13, 14, 15, 16], but we shall not need them in this work.

	t e e e	t t e e	t t t e	t t t t
e e e e (<i>TTTT</i>)		TTTTTE	TTTTTT	TTTTTT
t e e e (<i>TTTE</i>)	TTTEEE			

Figure 1. The four causal classes of Newtonian frames. Roman letters (e, t), capital letters (E, T), calligraphic (\mathcal{E}, \mathcal{T}) and Italic (e, t) letters represent the causal orientations (spacelike, timelike) respectively of the vectors of the frame, of their associated 2-planes, of their associated 3-planes and of the covectors of the coframe. This causal classification extends naturally to coordinate systems in causal homogeneous regions.

In particular, given a Newtonian frame $\{v_1, v_2, v_3, v_4\}$, a covector θ^α of its *dual* frame $\{\theta^1, \theta^2, \theta^3, \theta^4\}$ is timelike (resp. spacelike) iff the 3-plane generated by $\{v_\beta\}_{\beta \neq \alpha}$ is spacelike (resp. timelike).

On account of the above considerations, the causal orientations of the four vectors of a Newtonian frame determine unambiguously the causal orientations of their six associated 2-planes and the causal orientations of their four associated 3-planes. Consequently, we reach the following result.

Theorem 2 *In the 4-dimensional Newtonian space-time there exist four, and only four, causal classes of frames.*

The four Newtonian causal classes are represented in Fig. 1 whose reading is as follows.

- (i) The first column shows the sets of causal orientations $c_A = \{e e e e\}$, $c_A = \{t e e e\}$ of the covectors of the coframe (or correspondingly, of the sets of causal orientations $\bar{c}_A = \{\mathcal{T} \mathcal{T} \mathcal{T} \mathcal{T}\}$, $\bar{c}_A = \{\mathcal{T} \mathcal{T} \mathcal{T} \mathcal{E}\}$ of the four 3-planes of the frame or of the four families of coordinate hypersurfaces of a coordinate system). As stated in Lemma 2, only these two sets are possible, up to permutations.
- (ii) The first file shows the sets of causal orientations $c_A = \{t e e e\}$, $c_A = \{t t e e\}$, $c_A = \{t t t e\}$, $c_A = \{t t t t\}$ of the vectors of the frames or, correspondingly, the sets of causal orientations of the congruences of coordinate lines of a coordinate system. As stated in Lemma 1, only four sets are possible, up to permutations.
- (iii) Each not empty (p, q) -cell ($p = 1, 2$; $q = 1, 2, 3, 4$) shows the set of causal orientations C_{AB} of the associated 2-planes of vectors of the q -th frame, that corresponds to the p -th coframe or, correspondingly, the set of causal orientations of the six coordinate surfaces of a coordinate system.
- (iv) Permutations of the vectors of the frame or of the covectors of the coframe induce permutations of the associated 2-planes and 3-planes, but do not alter their causal

class. Correspondingly, permutations of the lines or hypersurfaces of a coordinate system induce permutations of the coordinate surfaces of the system, but do not alter its causal class.

For instance, standard frames, i.e. those that are locally realized with three rods and one clock at rest with respect to the rods, belong to the causal class $\{teee, TTTEEE, teee\}$. The history of the clock is a timelike coordinate line. The other coordinate lines are spacelike straight lines tangent to the rods at every (clock's) instant.

Geometrically, this causal class is better visualized by the family of spacelike instantaneous 3-planes generated by the directions of the three rods and the three families of timelike 3-planes (each one being the history of the 2-plane generated by two rods), whose normals or algebraic duals define the natural coframe $\{teee\}$. The mutual cuts of these coordinate 3-planes give the six families of coordinate 2-planes (denoted $\{TTTEEE\}$, three of them being timelike and the other three ones being spacelike). The coordinate planes cut in four congruences of coordinate lines (now denoted $\{teee\}$, one being timelike and the others being spacelike).

As already mentioned, the simplicity of the Newtonian causal structure with respect to the Lorentzian one lies in that the causal type of a Newtonian frame determines completely its causal class. This is related to the fact that, in Newtonian space-time, any set of spacelike vectors always generates a spacelike subspace. As a consequence, the number of causally different Newtonian classes of frames is equal to the dimension of the space. This is a general property, independent of the dimension n of the space-time. Denoting by $\{k\,t, (n-k)\,e\}$ the causal type of a basis with k timelike vectors and $n-k$ spacelike ones*, we therefore have:

Theorem 3 *In the n -dimensional Newtonian space-time there exist n causal classes of frames. A basis whose causal type is $\{k\,t, (n-k)\,e\}$, $k = 1, \dots, n$, has $\binom{n-k}{r}$ spacelike associated r -planes and $\binom{n}{r} - \binom{n-k}{r}$ timelike associated r -planes ($r = 1, \dots, n$).*

In dimension n , the causal classification of Newtonian frames in n classes induces a causal partition of the general lineal group $GL(n)$. Like in the Lorentzian case, the restriction of $GL(n)$ to a sole of these partitions simplifies notably the study of *intrinsic* deformations or perturbations of metric structures. In other, more intuitive, words, when one performs an arbitrary deformation of a metric structure, one obtains a mixed result: a wanted variation of the metric structure itself and a superfluous variation of the fields of frames (gauge) with respect to which the metric is expressed. Our causal classification allows us to reduce the group of deformations by considering its “quotient” by the causal classes, that is to say, roughly speaking, by considering nothing but the n -th part of the group which transforms metric structures but respects the causal class of the field of frames in which they are expressed. But this aspect will be analysed elsewhere.

* The comma between different causal orientations is put in this condensed expression only for visual clarity.

In what follows, we will construct some examples of transformations of $GL(n)$ that change the causal class of a starting coordinate system and also we will give direct examples of coordinate systems of the unusual causal classes. But previously we need to specify some simple but important notions.

4. Coordinate parameters, gradient coordinates and synchronizations

Whatever be the complete description of a coordinate system, it may be equivalently determined by its coordinate hypersurfaces, that is to say, by the four one-parameter families of hypersurfaces whose mutual cuts give the six families of coordinate surfaces, which in turn cut in the four congruences of coordinate lines.

Conversely, when the coordinate system is already known, say $\{x^\alpha\}$, these geometric elements may be easily discerned: the four one-parameter families of coordinate hypersurfaces are given by $\{x^\alpha = \text{constant}\}$, the six two-parameter families of coordinate surfaces are given by $\{x^\alpha = \text{constant}, x^\beta = \text{constant}\}$, and the four three-parameter families of coordinate lines are given by $\{x^\alpha = \text{constant}, x^\beta = \text{constant}, x^\gamma = \text{constant}\}$ for superscripts α, β, γ such that $\alpha \neq \beta \neq \gamma \neq \alpha$.

What Fig. 1 shows is nothing but the four possibilities of causal orientation of these geometric elements in Newtonian space-time. Thus, for example, the class $\{\text{ttte}, \text{TTTTT}, \text{eeee}\}$ represents those coordinate systems whose four coordinate hypersurfaces are all timelike $\{\mathcal{T}\mathcal{T}\mathcal{T}\mathcal{T}\}$, cut in six families of timelike coordinate surfaces $\{\text{TTTTT}\}$, which in turn cut in four congruences of coordinate lines $\{\text{ttte}\}$, three of them timelike and the other one spacelike.

4.1. Coordinate parameters and gradient coordinates

In fact, in any space-time, every coordinate x^α plays two extreme roles: that of a (coordinate) hypersurface for every constant value, of gradient dx^α , and that of a (coordinate) line when the other coordinates remain constant, of tangent vector ∂_α . This simple fact shows that, in spite of our deep-seated custom of associating to a coordinate a causal orientation, saying that *it is* timelike, lightlike or spacelike, *this appellation is not generically coherent*. Causal orientations are generically associated with directions or sets of directions of geometric objects, but not with space-time variables or parameters associated to them. In the case of a coordinate x^α , this generic incoherence appears because its two natural variations in the coordinate system, dx^α and ∂_α , have *generically different* causal orientations. *Only* when both causal orientations coincide, it is conceptually clear to extend to x^α itself the appellation of the common causal orientation of its two mentioned variations.

Consequently, we shall say generically of a coordinate x^α that it is a c_α *gradient coordinate* and a c_α *coordinate parameter* when the causal orientations of its variations dx^α and ∂_α be respectively c_α and c_α .

In addition, of a coordinate t which is a timelike coordinate parameter and a

timelike (resp. spacelike) gradient coordinate, we shall say also that it defines a *spacelike* (resp. *timelike*) *synchronization* (the coordinate hypersurfaces $t = \text{constant}$ being the synchronous event loci of the coordinate lines $t = \text{variable}$. See below).

It is to be noted that the appellation “timelike coordinate parameter” in place of the usual “timelike coordinate” when t is also a timelike synchronization is the correct one, because in that case t may be a constant or even a decreasing parameter along future oriented timelike trajectories of the space-time coordinate region, an odd property for a “time coordinate”.

A paradigmatic example of this situation is the oldest timelike coordinate parameter known by humanity, the *local Solar time*, that will be considered in Section 5. But before analyzing it, it is worthwhile to first present the group of (pure) synchronizations and its finite dimensional subgroup, the group of (pure) linear synchronizations.

4.2. The Synchronization Group

Consider a set of clocks in some region of a space-time. Their histories constitute a set of timelike lines on the region, naturally parameterized by the time t of the clocks. A *synchronization* is the stipulation of the locus of events where the clocks display the time $t = t_0$ for some chosen constant value t_0 .

We are interested here for ‘smooth situations’, in which the smallness of the clocks, their number and their histories are such that they can be efficiently described by a (sufficiently differentiable) congruence of timelike lines, $\gamma(t)$, and for which the locus of events $t = t_0$ defining the synchronization constitute a (sufficiently differentiable, transverse) hypersurface, $\varphi(x) = t_0$. Once the trajectories so synchronized, the loci of events $t = \text{constant}$ for *any* constant define a one-parameter family of hypersurfaces, to which the initial hypersurface $\varphi(x) = t_0$ belongs; let $\varphi(x) = t$ be its equation.

Any of these hypersurfaces $\varphi(x) = t$ is said to define the *same* synchronization that the hypersurface $\varphi(x) = t_0$. Denoting by $\dot{\gamma}$ the tangent vector to the histories of the clocks, $\dot{\gamma} \equiv \frac{d}{dt}\gamma(t)$, such space-time function $\varphi(x)$ verifies $\mathcal{L}(\dot{\gamma})\varphi = 1$, where $\mathcal{L}(\dot{\gamma})$ is the Lie derivative[‡] with respect to $\dot{\gamma}$.

Conversely, it is easy to see that the level hypersurfaces $\psi(x) = k$, $k = \text{constant}$, of any function $\psi(x)$ that verifies $\mathcal{L}(\dot{\gamma})\psi = 1$, define a synchronization for the (congruence of histories of the) clocks, i.e. there exists a canonical parameter t for the field $\dot{\gamma}$, $\frac{d}{dt}\gamma(t) = \dot{\gamma}$, such that $k = t$.

Consequently, for a congruence of (histories of) clocks of tangent vector field $\dot{\gamma}$, *the set of all its possible synchronizations is the set of all the scalar functions $\psi(x)$ such that $\mathcal{L}(\dot{\gamma})\psi = 1$* . And it is obvious that, if φ is such a synchronization, *any* other synchronization ψ is of the form $\psi = \varphi + \omega$, where ω is an invariant function of the field $\dot{\gamma}$, $\mathcal{L}(\dot{\gamma})\omega = 0$. The group of transformations of (pure) synchronizations for the congruence of clocks, or *synchronization group*, is thus isomorphic to the additive group of functions

[‡] On functions φ the Lie derivative reduces to a directional derivative, $\mathcal{L}(\dot{\gamma})\varphi = \dot{\gamma}(d\varphi) = \dot{\gamma}^\rho \partial_\rho \varphi$.

$\{\omega\}$ which are invariant for the congruence $\dot{\gamma}$: if φ is an initial synchronization and ω any $\dot{\gamma}$ -invariant function, any other synchronization ψ is obtained by $\psi = T_\omega \varphi \equiv \varphi + \omega$.

To make more explicit the synchronization group as a transformation group of the space-time, let us start from a coordinate system $\{x^\alpha\}$ ($\alpha = 0, 1, \dots, n-1$) adapted both, to the field $\dot{\gamma}$, say $\dot{\gamma} = \partial_0$, and to the synchronization φ , thus $d\varphi = dx^0$. In this coordinate system, the $\dot{\gamma}$ -invariant character of a function ω is expressed by its independence of the timelike coordinate parameter x^0 , $\omega = \omega(x^i)$, ($i = 1, \dots, n-1$). The new coordinate system $\{X^\alpha\}$, generated by ω and adapted both to $\dot{\gamma}$ and to $T_\omega \varphi = \psi$ is then of the form

$$X^0 = x^0 + \omega(x^i), \quad X^i = x^i. \quad (1)$$

These are the space-time transformation equations of the synchronization group.

For our purpose here, that of generating easily the Newtonian causal classes, it is nevertheless sufficient to consider the simplest subgroup of the synchronization group (1), the *linear synchronization group*:

$$X^0 = x^0 + a_i x^i, \quad X^i = x^i. \quad (2)$$

Its matrix form may be analyzed as follows. Let $\mathbf{1}$ be the $n \times 1$ column matrix of components $(1, 0, \dots, 0)$, and consider the set of all the $1 \times n$ matrices \mathbf{a} orthogonal to $\mathbf{1}$, $\mathbf{a} \cdot \mathbf{1} = 0$; they are obviously of the form $\mathbf{a} = (0, \vec{a})$ with $\vec{a} \equiv (a_1, \dots, a_{n-1})$. Then, the *linear synchronization algebra* is the (commutative) algebra of matrices of the form $\mathbf{1} \otimes \mathbf{a}$, so that the matrices L of the linear synchronization group are of the form $L = \exp\{\mathbf{1} \otimes \mathbf{a}\} = I + \mathbf{1} \otimes \mathbf{a}$, which clearly correspond to matrices of minimal polynomial $(L - I)^2 = 0$. In obvious matrix notation, equations (2) may be written $\mathbf{X} = L\mathbf{x}$.

From equations (2) we have the relations between the natural frames and coframes of two coordinate systems related by a linear synchronization:

$$\partial_{X^0} = \partial_{x^0}, \quad \partial_{X^i} = -a_i \partial_{x^0} + \partial_{x^i}, \quad (3)$$

$$dX^0 = dx^0 + a_i dx^i, \quad dX^i = dx^i. \quad (4)$$

Remark that, until now, all the considerations about the synchronization group remain valid for both, Newtonian and relativistic space-times and are applicable to *any* starting coordinate system.

5. Examples of Newtonian coordinate systems of different causal classes

5.1. Generating Newtonian causal classes by the Linear Synchronization Group

Surprisingly enough, the linear synchronization group provides one of the simplest ways of generating *all* the Newtonian causal classes.

In what follows, we will always start, in the Newtonian space-time, from a *standard coordinate system* $\{x^\alpha\}$, that is to say a coordinate system such that the coordinate lines $x^0 = t$, $x^i = \text{constant}$ are synchronized by the instantaneous spaces of the absolute time current θ , $dx^0 = \theta = dt$, and such that the other coordinate lines $x^i = \text{variable}$

are tangent to these instantaneous spaces, $\gamma^*(\partial_i) = 0$. Its natural frame is thus of the causal type $\{te \dots e\}$.

Let us apply the transformation (2) to this coordinate system. By construction (definition of a change of synchronization) the new coordinate X^0 is a timelike coordinate parameter, because ∂_{X^0} is the expression, in this coordinate system $\{X^\alpha\}$, of $\dot{\gamma}$, which is timelike. However, X^0 results to be a spacelike gradient coordinate whenever $\vec{a} \neq 0$, because then, according to (4), one has $dX^0 \wedge dt \neq 0$. On the other hand, every new coordinate X^i is a timelike coordinate parameter whenever the corresponding component a_i of \vec{a} does not vanish, because ∂_{X^i} , which is given by the second of expressions (3), is timelike in this case, $\gamma^*(\partial_{X^i}) \neq 0$. Nevertheless X^i remains a spacelike gradient coordinate, because $\forall i, dX^i \wedge dt \neq 0$.

We see thus that, in the n -dimensional Newtonian space-time, starting from a standard coordinate system $\{t, x^i\}$ of causal type $\{t, (n-1)e\}$, the linear synchronization transformations (2) for every one of the vectors $\vec{a} = (1, \vec{a}, 1, 0, \vec{a}, 0)$, ($k = 1, \dots, n$), define a coordinate system $\{X^\alpha\}$ of causal type $\{kt, (n-k)e\}$, belonging to the k -th causal class of the n possible ones, according to theorem 3. Then, for every $r = 1, \dots, n$, the $\binom{n}{r}$ associated r -planes are of causal type $\{[(\binom{n}{r} - \binom{n-k}{r})T, \binom{n-k}{r}E]\}$.

For $n = 4$, this gives of course the four causal classes of Figure 1.

It is worthwhile to note that all the different causal classes have been obtained by simple, *pure*, changes of synchronization of the *same* system of clocks, excluding any other change of coordinates or of observers. Apparently, this is not an intuitive idea for most of us.

5.2. The causal class of the ancestral local Solar time

The local Solar time, i.e. the time shown by a sundial, is the oldest timelike coordinate parameter known by humanity, and still remains indefinitely alive and currently in use, although slightly deformed by the at present stepped time zones. As we have already mentioned, this local Solar time is a paradigmatic example of the situations where the current but particular notion of “timelike coordinate” becomes incoherent.

Specifically, we will consider here the causal class of a coordinate system at rest with respect to a spherical Earth in uniform rotation when the (absolute time rythmed) clocks are synchronized by the *local Solar time or sundial synchronization*, i.e. are such that at any place they watch the same fixed time (say 12h) when the Sun is just on the local meridian. For simplicity, we have not taken into account the inclination of the ecliptic and have neglected the translational motion of the Earth.

Let $\{t, r, \theta, \phi\}$ be a standard coordinate system where $\{r, \theta, \phi\}$ are the usual geocentric inertial spherical coordinates. This system thus belongs to the standard causal class $\{teee, TTTEEE, teee\}$.

The *geocentric rotating spherical coordinate system* $\{t, r, \theta, \Phi\}$, is obviously given by the (pure) rotation

$$\Phi = \phi - \omega t, \quad (5)$$

where ω is the Earth's angular velocity. Here the coordinate lines where only t varies are no longer inertial, but the timelike helices that they describe remain synchronized by the instantaneous spaces of the time current. This point, and the fact that the sole new coordinate Φ verifies $d\Phi \wedge dt \neq 0$, make the causal class of this rotating coordinate system to remain the standard one.

Now, starting from this coordinate system $\{t, r, \theta, \Phi\}$, let us perform a (pure) synchronization change of the form (2) to the *Solar time geocentric rotating spherical*

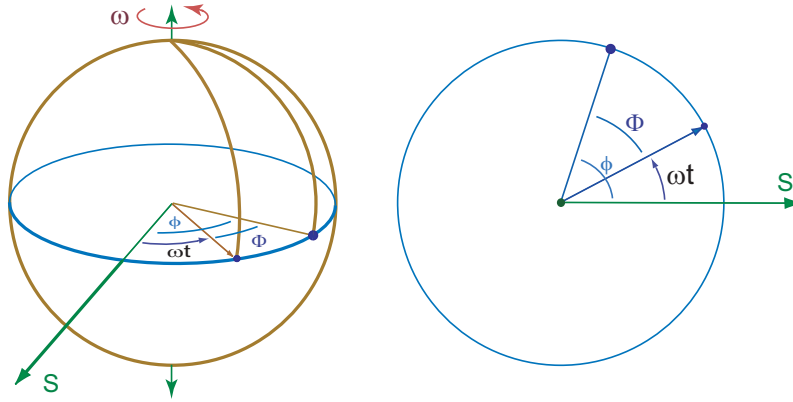


Figure 2. The geocentric inertial spherical standard coordinates $\{t, r, \theta, \phi\}$ and the local Solar time geocentric rotating spherical coordinates $\{T, r, \theta, \Phi\}$ are related by $T = \frac{\phi}{\omega}$, $\Phi = \phi - \omega t$, where ω is the angular velocity of the Earth. The fixed direction S is that of the sun (the inclination of the ecliptic is not taken into account and the translational motion of the Earth is neglected). The picture on the right shows the Earth equator, $r = R_{\oplus}$, $\theta = 0$, whose history in the plane $\{T, \Phi\}$ is represented in Fig. 3.

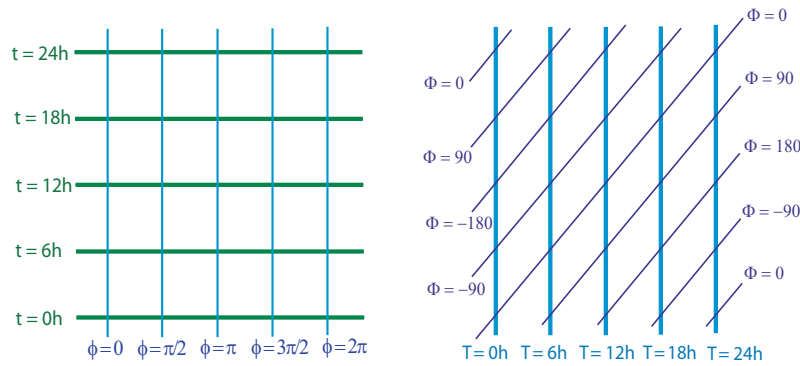


Figure 3. History of the Earth equator $r = R_{\oplus}$, $\theta = 0$ in the plane $\{T, \Phi\}$. (a) In geocentric inertial spherical coordinates: the vertical thin straight lines are coordinate lines of the absolute time t , and the horizontal thick straight lines correspond to the absolute synchronization (hypersurfaces of simultaneity $t = \text{constant}$). (b) In an Earth rotating frame: the histories of the equator events, which constitute the coordinate lines of the 'solar time' T , are represented by the inclined thin straight lines $\Phi \equiv \phi - \omega t = \text{constant}$, meanwhile the 'solar synchronization' hypersurfaces $T = \text{constant}$ are represented by the vertical thick straight lines. Note that this 'solar instants' contain the coordinate lines of the absolute time $t = \text{variable}$.

coordinate system $\{T, r, \theta, \Phi\}$, that is to say,

$$T = t + \frac{\Phi}{\omega} = \frac{\phi}{\omega}. \quad (6)$$

In this Solar time rotating coordinate system (see Fig. 2), the observers at rest with respect to the rotating Earth remain at rest by construction, although their local time T has been synchronized to be given by a sundial for a fixed Sun placed in the initial system at the meridian of longitude $\phi = 0$ (see Fig. 3). So, the coordinate lines where only T varies, of tangent vector ∂_T , are timelike.

On the other hand, the coordinate lines where only Φ varies, of tangent vector ∂_Φ , are also timelike, because the inverse transformation is, from (6), $\{t = T - \Phi/\omega, r, \theta, \phi = \omega T\}$ and it follows $\omega\partial_\Phi = -\partial_t$: they form, in fact, the congruence of inertial observers, as it is due. Although we could compute the causal character of the corresponding 1-forms, surfaces and hypersurfaces, this is not necessary because the table of Fig. 1 gives already this information. Thus, *the causal class of the ancestral local Solar time coordinate system is $\{\text{tee}, \text{TTTTTE}, \text{eeee}\}$.*

It is worthwhile to note that, in Newtonian physics as well as in relativity, the more natural and ancestral synchronization is generated by *timelike* hypersurfaces, a fact that seems systematically forgotten in theoretical physics, where a synchronization is always defined by *spacelike* hypersurfaces.

5.3. Newtonian Emission Coordinates

Suppose an inertial medium in which a class of signals (sound, light) propagates at constant velocity v . Let $\kappa(t)$ be the space-time point-like trajectory of an emitter clock that uses such signals to continuously broadcast his time t . In the space-time, the front waves describe thus (sound-, light-)cones carrying the value $t = \text{constant}$. Four such emitters $\kappa^A(t)$ ($A = 1, 2, 3, 4$) fill the space-time with four (one-parameter) families of cones $t^A = \text{constant}$ which generically define a space-time system of *emission coordinates*.

Let us take every event as the vertex of the past cone of the velocity v of propagation of the class of signals in question. This cone cuts the four histories $\kappa^A(t)$ of the clocks at the clock times t^A . Then, the set $\{t^A\}$ constitutes the four emission coordinates of the event.

Here we will consider the simple case of four emitters at rest with respect to the inertial medium referred to a standard coordinate system $\{t, x^i\} = \{t, \vec{r}\}$, of worldlines

$$\kappa^A(t) = (t, \vec{c}^A). \quad (7)$$

Then, the signal emitted by the clock κ^A at the instant t^A at velocity v describes in the space-time a cone of equation

$$v(t - t^A) = |\vec{r} - \vec{c}^A|, \quad (8)$$

so that the emission coordinates $\{t^A\}$ are related to the inertial ones $\{t, \vec{r}\}$ by

$$t^A = t - \frac{1}{v} |\vec{r} - \vec{c}^A|. \quad (9)$$

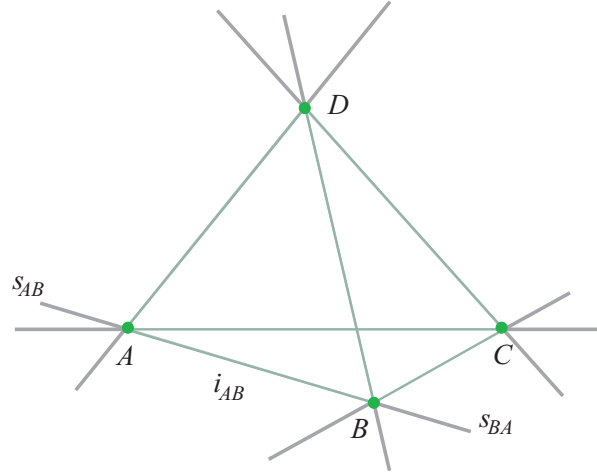


Figure 4. At any instant $t = \text{constant}$, the positions $\kappa^A(t) \equiv A$ ($A = 1, 2, 3, 4$) of the four clocks generically define the four vertices A, B, C, D (all \neq) of a 3-dimensional tetrahedron. If the clocks are at rest in an inertial system, the outer open segments s_{AB} and s_{BA} of the straight line ℓ_{AB} containing the edge i_{AB} between the vertices A and B represent the shadows of the signals B and A respectively produced by A and B .

To know the causal class of the emission coordinates $\{t^A\}$ it is convenient to consider the coordinate r -forms. From (9), the coframe of 1-forms $\{dt^A\}$ may be written

$$dt^A = dt + \omega^A, \quad \omega^A \equiv -\frac{1}{v}u^A, \quad (10)$$

where u^A is the 1-form associated to the generically unit spacelike vector \vec{u}^A , given by

$$\vec{u}^A \equiv \frac{\vec{r} - \vec{c}^A}{|\vec{r} - \vec{c}^A|}, \quad (11)$$

$u^A = \gamma(\vec{u}^A)$, γ being the 3-dimensional inverse of the structure metric γ^* associated to the inertial observers ∂_t , $\gamma.\gamma^* = I - \theta \otimes \partial_t$, and θ being the time current^{††}. The Jacobian matrix of the transformation (9) is not defined at the events (t, \vec{r}) where $\vec{r} = \vec{c}^A$, that is to say, along the clock worldlines κ^A . Below we shall see other events where the Jacobian matrix is not defined. Out of these worldlines one has $\omega^A \neq 0$ and thus dt^A is spacelike (it is not collinear to the time current). Consequently, *the coframe of the Newtonian emission coordinate system is of causal type $\{e e e e\}$.*

The co-planes of the coordinate system are determined by the 2-forms

$$dt^A \wedge dt^B = dt \wedge (\omega^B - \omega^A) + \omega^A \wedge \omega^B, \quad (12)$$

so that the co-plane AB is generically spacelike, and can be timelike only when $\omega^A \wedge \omega^B = 0$, that is to say on the timelike plane of events Π_{AB} that contains the

^{††}Note that, meanwhile γ^* is an intrinsic element of the geometry of Newtonian space-time, its ‘three-dimensional inverse’ γ is an *observer-dependent* quantity, given by $\gamma.\gamma^* = I - \theta \otimes u$, where u is the unit velocity of the chosen observer. To two different observers, they correspond two different degenerate four-dimensional covariant metrics γ of rank three, although their induced spatial components on the instantaneous space take the same value, as it is well experienced in the usual three-dimensional formalism.

worldlines of κ^A and κ^B . Because the clocks are at rest with respect to the starting inertial system, at any $t = \text{constant}$ their positions $\kappa^A(t) \equiv A$ will generically define the four vertices A, B, C, D (all \neq) of a 3-dimensional tetrahedron (see Figure 4). Denote by ℓ_{AB} the straight line passing through A and B and, in it, by i_{AB} the corresponding open edge of the tetrahedron and by s_{AB} (*respect. s_{BA}*) the other open segment contiguous to A (*respect. contiguous to B*). It is then clear that the timelike plane Π_{AB} is the history of the straight line ℓ_{AB} , and we will denote by I_{AB} the history of i_{AB} , the (timelike) open strip of Π_{AB} whose boundaries are κ^A and κ^B . Similarly, S_{AB} (*respect. S_{BA}*) will denote the (timelike) open strip of Π_{AB} contiguous to κ^A (*respect. contiguous to κ^B*). Now we see that the condition $\omega^A \wedge \omega^B = 0$ takes place along ℓ_{AB} , thus on the events of Π_{AB} . In addition, because from (10) all the ω^A have same length, one has $\omega^A = -\omega^B$ on i_{AB} , thus on the events of I_{AB} , and $\omega^A = \omega^B$ on the two other open segments s_{AB} and s_{BA} , thus of the events of S_{AB} and S_{BA} , where one has

$$dt^A \wedge dt^B = 0, \quad (13)$$

and the coordinate system degenerates. These open strips of Π_{AB} , S_{AB} and S_{BA} , are also the half-planes describing the history of the *shadows* that the clocks A and B make respectively to the signals of the clocks B and A . These considerations on expressions (12) and (13) show that either all the coordinate coplanes are spacelike, or one of them is timelike, so that, on account of lemma 3, it results that *generically the type of the coordinate planes is $\{\text{T T T T T T}\}$ but on the events of the six timelike strips I_{AB} , and only on them, the type is $\{\text{T T T T T E}\}$, the coordinate system being degenerate on the shadows S_{AB} and S_{BA} and undetermined on the worldlines κ^A .*

To analyze the coordinate lines, let us consider the dual 3-forms:

$$\begin{aligned} dt^A \wedge dt^B \wedge dt^C &= \omega^A \wedge \omega^B \wedge \omega^C \\ &+ dt \wedge (\omega^A \wedge \omega^B + \omega^B \wedge \omega^C + \omega^C \wedge \omega^A). \end{aligned} \quad (14)$$

The 3-coplane ABC is generically spacelike, and can be timelike only when $\omega^A \wedge \omega^B \wedge \omega^C = 0$, what happens on the events of the timelike 3-plane Π_{ABC} that contains the worldlines κ^A , κ^B , κ^C . In the stationary 3-dimensional sections $t = \text{constant}$, these events correspond to the planes ℓ_{ABC} that contain the three clocks A, B, C , and thus the three lines ℓ_{AB} , ℓ_{BC} , ℓ_{CA} , including the tetrahedral faces i_{ABC} that their edges i_{AB} , i_{BC} and i_{CA} delimit, and the six strips s_{AB} , s_{BA} , s_{BC} , s_{CB} , s_{CA} , s_{AC} . We already know that, apart from on the clocks A, B, C themselves, on these last six strips the coordinate coplanes degenerate; are there other events than that on which the coordinate 3-coplanes be degenerate? In other words, there where $\omega^A \wedge \omega^B \wedge \omega^C = 0$ out of the edges, can the other term in (14) also vanish? We have:

$$\omega^C = \alpha \omega^A + \beta \omega^B, \quad (15)$$

so that (14) becomes

$$dt^A \wedge dt^B \wedge dt^C = (1 - \alpha - \beta) dt \wedge \omega^A \wedge \omega^B, \quad (16)$$

which cannot degenerate, being $\omega^A \wedge \omega^B \neq 0$, unless $\alpha + \beta = 1$. But

$$1 = (\omega^C)^2 = \alpha^2 + \beta^2 + 2\alpha\beta(\omega^A \cdot \omega^B) = 1 + 2\alpha\beta(\omega^A \cdot \omega^B - 1), \quad (17)$$

admits no solution, because $\alpha \neq 0 \neq \beta$ and necessarily $\omega^A \cdot \omega^B < 1$. The tangent vectors to the coordinate lines being at every event causally related to the 3-planes by lemma 3, it results the following.

The coordinate lines of the emission coordinates in Newtonian space-times are generically of type {tttt}, but on the events of the timelike 3-planes Π_{ABC} containing three emitters they are generically of type {ttte}, and are of type {ttee} on the events of the timelike strips I_{AB} generated by every pair of clocks.

It is pertinent here to note that, in Newtonian space-time, the emission coordinate system generated by a positioning system is never causally homogeneous, but always presents three regions corresponding to the non standard three causal classes. Only the emission coordinate systems generated by relativistic positioning systems based in light signals are always causally homogeneous, as we will see in next section.

The geometry of the coordinate surfaces and coordinate lines of the emission coordinates is simple. Because generated by the two by two or three by three intersections of the coordinate hypersurfaces, which are isotropic cones of parallel axes, *the coordinate surfaces and coordinate lines of the emission coordinates are hyperboloids and hyperbolas respectively.* As already seen, these hyperbolas are generically timelike lines, up to at their base point, where they become spacelike.

As we have seen, the transformation (9) from a standard inertial coordinate system $\{t, x^i\} = \{t, \vec{r}\}$ to an emission coordinate system $\{t^A\}$ is degenerate on the clock shadows S_{AB} , timelike space-time surfaces generated by every clock for the signals coming to it from the others. Thus the question: is transformation (9) degenerate at other events than those of the shadows S_{AB} ? To see it, let us consider the coordinate volume element η_{ec} :

$$\begin{aligned} \eta_{ec} &\equiv dt^A \wedge dt^B \wedge dt^C \wedge dt^D \\ &= dt \wedge [-\omega^A \wedge \omega^B \wedge \omega^C + \omega^B \wedge \omega^C \wedge \omega^D \\ &\quad - \omega^C \wedge \omega^D \wedge \omega^A + \omega^D \wedge \omega^A \wedge \omega^B] \\ &= -dt \wedge [(\omega^A - \omega^D) \wedge (\omega^B - \omega^D) \wedge (\omega^C - \omega^D)] . \end{aligned} \tag{18}$$

It is then evident that the Jacobian is degenerate, as we already know, there where $\omega^A = \omega^B$, that is to say, on the clock shadows S_{AB} , for any pair $A \neq B$. But (18) shows that it can be also degenerate there where the three vectors $\omega^A - \omega^D$ are linearly dependent. It can be seen (for example in [2]), that this happens on the events where the signals coming from the four clocks are seen or heard as coming from four points located on a circle of the celestial sphere of the event (quotient of the instantaneous space of the event by the radial distance to the event).

6. Lorentzian causal classes with Newtonian analogues

Theorem 1 establishes the existence of 199 Lorentzian causal classes of space-time frames [1]. Among them, one can found the analogue to the 4 Newtonian causal classes of space-time frames [2] i.e., four Lorentzian classes of frames having the same causal signature that the four Newtonian ones. Thus, whatever be the relativistic space-time,

one can always choose local coordinate systems belonging, in some region, to any of the 199 causal classes and, in particular, having the same causal signature that any given Newtonian coordinate system. But it must be emphasized that going from 4 to 199 causal classes, the change from the Newtonian conception of the space-time to the relativistic one implies a richness of causally different ways of locating space-time events that, in spite of the appearances extracted from the current scientific publications, is far from being well understood.

Here, we shall analyze in Minkowski space-time the situations that we have already analyzed in the Newtonian case.

6.1. The Linear Synchronization Group

Let us consider, in Minkowski space-time, the linear synchronization group (2) acting on an inertial laboratory referred to a standard coordinate system $\{x^0, x^i\}$. The metric components $\eta_{\alpha\beta}$ of the Minkowski flat metric η in this coordinate system are the usual $\eta_{\alpha\beta} = \text{diag}(-1, 1, \dots, 1)$ so that the associated natural frame is of the causal type $\{\text{t e} \dots \text{e}\}$.

The natural frame and coframe of the new system $\{X^\alpha\}$ are given by (3) and (4). It follows, by direct scalar products of these expressions, that the covariant and contravariant components, $g_{\alpha\beta}$ and $g^{\alpha\beta}$ respectively, of the metric η in this new system are:

$$g_{\alpha\beta} = \begin{pmatrix} -1 & \vec{a} \\ \vec{a} & I - \vec{a} \otimes \vec{a} \end{pmatrix}, \quad g^{\alpha\beta} = \begin{pmatrix} -1 + \vec{a}^2 & \vec{a} \\ \vec{a} & I \end{pmatrix}. \quad (19)$$

where $\vec{a} \equiv (a_1, \dots, a_{n-1})$, $\vec{a}^2 \equiv \sum_{i=1}^{n-1} a_i^2$ and I is the $n-1$ identity matrix.

We can see from (19) that, *like in the Newtonian case, the new coordinate X^0 is a timelike coordinate parameter*. However, X^0 results to be a *spacelike gradient coordinate only when $|\vec{a}| > 1$, meanwhile in the Newtonian case the condition is simply $\vec{a} \neq 0$* . When $|\vec{a}| = 1$ or $|\vec{a}| < 1$, X^0 is a *null or timelike gradient coordinate, respectively*. Obviously, the first of these last two situations is forbidden in the Newtonian case, and the second one cannot be attained by the linear synchronization group (up to, trivially, by the identity transformation, $\vec{a} = 0$).

On the other hand, *every new coordinate X^i remains, like in the Newtonian case, a spacelike gradient coordinate*. However X^i results to be a *timelike coordinate parameter only when $|a_i| > 1$, meanwhile in the Newtonian case the condition is simply $a_i \neq 0$* . When $|a_i| = 1$ or $|a_i| < 1$, X^i is a *null or spacelike coordinate parameter, respectively*. Both situations are also absent in the Newtonian case (up to for $\vec{a} = 0$).

Finally, the coordinate two-forms satisfy:

$$dX^i \wedge dX^j = dx^i \wedge dx^j, \quad dX^0 \wedge dX^i = dx^0 \wedge dx^i + a_j dx^i \wedge dx^j, \quad (20)$$

$$(dX^i \wedge dX^j)^2 = 1, \quad (dX^0 \wedge dX^i)^2 = -1 + \vec{a}^2 - a_i^2. \quad (21)$$

Consequently, the $(n-2)$ -coordinate surfaces $X^i = \text{constant}$, $X^j = \text{constant}$ (i, j given) are timelike and the $(n-2)$ -coordinate surfaces $X^0 = \text{constant}$, $X^i = \text{constant}$ (i given)

are timelike, null or spacelike if $\vec{a}^2 - a_i^2$ is greater, equal or smaller than 1, respectively. This information, insufficient for $n > 4$, completely determines the causal class of the coordinate system $\{X^0, X^i\}$ in the four-dimensional Minkowski space-time:

All the causal classes obtained by a linear synchronization transformations have a causal signature of the form:

$$\{t\ c_1\ c_2\ c_3, \ T\ T\ T\ C_{12}\ C_{13}\ C_{23},\ c_0\ e\ e\ e\} \quad (22)$$

where the non-fixed causal orientations, $c_1, c_2, c_3, C_{12}, C_{13}, C_{23}, c_0$ depend on the a_i parameters as follows:

$$c_i = \begin{cases} t & |a_i| > 1 \\ l & |a_i| = 1 \\ e & |a_i| < 1 \end{cases} \quad C_{ij} = \begin{cases} T & a_i^2 + a_j^2 > 1 \\ L & a_i^2 + a_j^2 = 1 \\ E & a_i^2 + a_j^2 < 1 \end{cases} \quad c_0 = \begin{cases} t & |\vec{a}| < 1 \\ l & |\vec{a}| = 1 \\ e & |\vec{a}| > 1 \end{cases} \quad (23)$$

A more detailed analysis of the compatible orientations shows that *the number of different causal classes that may be generated by a linear synchronization transformation is 29, in contrast with the only 4 Newtonian ones.* We will consider them elsewhere [17].

Evidently the four Newtonian analogues exist in relativity. In fact, *the Lorentzian causal classes of same causal signature that the four Newtonian ones correspond to the following values of the parameters a_i :*

$$\begin{aligned} \{t\ t\ t\ t, \ T\ T\ T\ T\ T\ T, \ e\ e\ e\ e\} & \quad \text{if} \quad \forall i, \quad |a_i| > 1 \\ \{t\ t\ t\ e, \ T\ T\ T\ T\ T\ T, \ e\ e\ e\ e\} & \quad \text{if} \quad \begin{cases} \exists! i, & |a_i| < 1 \\ \forall j \neq i, & |a_j| > 1 \end{cases} \\ \{t\ t\ e\ e, \ T\ T\ T\ T\ T\ E, \ e\ e\ e\ e\} & \quad \text{if} \quad \begin{cases} \exists! i, & |a_i| > 1 \\ j, k \neq i, & a_j^2 + a_k^2 < 1 \end{cases} \\ \{t\ e\ e\ e, \ T\ T\ T\ E\ E\ E, \ t\ e\ e\ e\} & \quad \text{if} \quad \forall i, \quad |a_i| < 1 \end{aligned}$$

6.2. The local Solar time synchronization.

In the Newtonian example of the rotating Earth of subsection 5.2, the latitude of the observer plays in fact no role, because we are interested only in the time synchronization, not in the angular height of the Sun. For this reason, we shall consider here, in place of the Earth, a rigidly rotating disk and, in place of spherical coordinates, cylindrical ones.

So, let $\{t, \phi, \rho, z\}$ be an inertial laboratory referred to a standard cylindrical coordinate system in Minkowski space-time. This coordinate system is known to belong to the standard causal class $\{t\ e\ e\ e, \ T\ T\ T\ E\ E\ E, \ t\ e\ e\ e\}$.

The *rotating cylindrical coordinate system* $\{t, \Phi, \rho, z\}$, adapted to the congruences of the observers in rigid rotation motion is defined by the transformation (5). In the Newtonian case this system remains in the standard class, as happens for the rotating spherical coordinate system considered in subsection 5.2.

As it is well known, in Minkowski space-time the light cylinder $\rho = 1/\omega$ generates other causal classes. Indeed, the covariant and contravariant components of the metric tensor in this rotating coordinate system are, respectively:

$$g_{\mu\nu} = \begin{pmatrix} -1 + \omega^2 \rho^2 & \omega \rho^2 & 0 & 0 \\ \omega \rho^2 & \rho^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -1 & \omega & 0 & 0 \\ \omega & \frac{1}{\rho^2} - \omega^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (24)$$

From here we easily obtain the following causal classes:

$$\{\text{teee}, \text{TTTEEE}, \text{teee}\} \quad \text{if } \rho < 1/\omega \quad (25)$$

$$\{\text{leee}, \text{TLEEE}, \text{leee}\} \quad \text{if } \rho = 1/\omega \quad (26)$$

$$\{\text{eeee}, \text{TEEEEE}, \text{teee}\} \quad \text{if } \rho > 1/\omega \quad (27)$$

The causal orientation c_α of the vectors of the coordinate frame is given by the sign of the diagonal elements $g_{\alpha\alpha}$ of the metric matrix; correspondingly, the causal orientation $C_{\alpha\beta}$ of the coordinate 2-surfaces is given by the signs of the second order diagonal minors, $g_{\alpha\alpha}g_{\beta\beta} - (g_{\alpha\beta})^2$; and finally the causal orientation c_α of the coordinate co-frame is given by the signs of the diagonal elements $g^{\alpha\alpha}$ of the inverse metric matrix $g^{\mu\nu}$.

Note that, *in the rotating system, t remains a timelike gradient coordinate*, which determines the events that are simultaneous with respect to the inertial observer at rest at the rotation axis. Nevertheless, *t only remains a timelike coordinate parameter in the interior of the light cylinder, $\rho < 1/\omega$.*

The timelike helices $t = \text{variable}$ are thus synchronized with an inertial time. But in the region $\rho \geq 1/\omega$ they become null or spacelike helices and they do not represent the history of a system of observers in rigid motion, as it is well known.

Now, starting from this rotating system $\{t, \Phi, \rho, z\}$, let us perform the Solar time linear synchronization change (6). In the new coordinate system $\{T, \Phi, r, \theta\}$, the covariant and contravariant components of the metric tensor are, respectively:

$$g_{\mu\nu} = \begin{pmatrix} -1 + \omega^2 \rho^2 & \frac{1}{\omega} & 0 & 0 \\ \frac{1}{\omega} & -\frac{1}{\omega^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} \frac{1}{\omega^2 \rho^2} & \frac{1}{\omega \rho^2} & 0 & 0 \\ \frac{1}{\omega \rho^2} & \frac{1}{\rho^2} - \omega^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (28)$$

From these coefficients, it then follows that, *in the interior $\rho < 1/\omega$ of the light cylinder, the Solar time rotating coordinate system $\{T, \Phi, r, \theta\}$ belongs to the causal class $\{\text{ttee}, \text{TTTTTE}, \text{eeee}\}$, of same causal signature that the Solar time geocentric rotating system of Newtonian space-time.*

On the light cylinder $\rho = 1/\omega$ the new coordinates belong to the causal class $\{\text{tlee}, \text{TTTLLE}, \text{leee}\}$ and on the exterior region $\rho > 1/\omega$ it becomes the standard class $\{\text{teee}, \text{TTTEEE}, \text{teee}\}$.

6.3. Relativistic Emission Coordinates

Let us consider now the relativistic analog of the emission coordinates defined in subsection 5.3. Now, every emitter κ is supposed to continuously broadcast, in an inertial non-dispersive medium, their proper time τ^A by means of sound or light signals that propagate in the medium at constant velocity $v \leq 1$.

As in subsection 5.3, the four emitters will be consider at rest with respect to the medium referred to a standard coordinate system $\{t, x^i\} = \{t, \vec{r}\}$. Then, the inertial time t is also the proper time of the four emitters and their worldlines take the expression (7): $\kappa^A(t) = (t, \vec{c}^A)$. Then, the equation of the cones that describe the signals is (8), and the emission coordinates $\{t^A\}$ are related to the inertial ones $\{t, \vec{r}\}$ by (9).

Let us first consider the (sound) case $v < 1$.

To know the causal class of the emission coordinate system $\{t^A\}$ we can start from the coframe of 1-forms $\{dt^A\}$ given in (10) and (11). Out of the clock worldlines κ^A , where the transformation (9) is not defined, dt^A is spacelike because:

$$(dt^A)^2 = -1 + \frac{1}{v^2} > 0 \quad (29)$$

Consequently, *the coframe of the relativistic emission coordinate system with $v < 1$ is of causal type $\{e e e e\}$.*

The co-planes of the coordinate system are determined by the 2-forms (12) that satisfy

$$(dt^A \wedge dt^B)^2 = -\frac{1}{v^4}(\mu_{AB}^2 - 2v^2\mu_{AB} + 2v^2 - 1), \quad \mu_{AB} \equiv u_A \cdot u_B. \quad (30)$$

Note that μ_{AB} is the cosine of the angle between the signals coming from the emitters A and B . The study of the polynomial (30) in μ_{AB} leads to the following: *the co-plane AB is spacelike, null or timelike according as μ_{AB} is greater, equal or smaller than $2v^2 - 1$.*

To analyze the coordinate lines, let us consider the dual 3-forms (14). We have:

$$(dt^A \wedge dt^B \wedge dt^C)^2 = \frac{1}{v^4} \left(\frac{1-v^2}{v^2} \Delta_D - \Lambda_D \right), \quad D \neq A, B, C, \quad (31)$$

where Δ_D and Λ_D depend on μ_{AB} as:

$$\Delta_D \equiv (u_A \wedge u_B \wedge u_C)^2 = 1 + 2\mu_{AB}\mu_{BC}\mu_{CA} - (\mu_{AB}^2 + \mu_{BC}^2 + \mu_{CA}^2) \quad (32)$$

$$\Lambda_D \equiv 2(1 - \mu_{AB})(1 - \mu_{BC})(1 - \mu_{CA}) \quad (33)$$

Thus, *the 3-coplane ABC is spacelike, null or timelike according as $\frac{\Lambda_D}{\Delta_D}$ is smaller, equal or greater than $\frac{1-v^2}{v^2}$.*

From this information it follows that *the causal classes of the emission coordinate systems $\{t^A\}$ are of the form:*

$$\{c_1 c_2 c_3 c_4, C_{12} C_{13} C_{14} C_{23} C_{24} C_{34}, e e e e\} \quad (34)$$

where the causal orientations, c_A , C_{AB} depend on the cosines μ_{AB} of the angles between the signals coming from the emitters A and B as:

$$c_A = \begin{cases} t & \frac{\Lambda_A}{\Delta_A} < \frac{1-v^2}{v^2} \\ 1 & \frac{\Lambda_A}{\Delta_A} = \frac{1-v^2}{v^2} \\ e & \frac{\Lambda_A}{\Delta_A} > \frac{1-v^2}{v^2} \end{cases} \quad C_{AB} = \begin{cases} T & \mu_{CD} > 2v^2 - 1 \\ L & \mu_{CD} = 2v^2 - 1 \\ E & \mu_{CD} < 2v^2 - 1 \end{cases} \quad (35)$$

with $C, D \neq A, B$. A more detailed analysis, which will be presented elsewhere [17], of the compatible orientations lead to the following result: *depending on the different configurations of the stationary emitters and/or of the different values of the velocity $v < 1$, the emission coordinate systems may present space-time regions of 102 different causal classes.*

It is worth mentioning that, *some emitters' configurations and sound velocities $v < 1$, generate space-time regions of the same causal signatures that those of the three Newtonian cases* (subsection 5.3).

Indeed, *the three Newtonian causal signatures are related to how the events receive the sound signals, according to the following three sets of conditions:*

$$\begin{aligned} \{tttt, TTTTTT, eeee\} & \quad \text{if} \quad \forall A, \quad \frac{\Lambda_A}{\Delta_A} < \frac{1-v^2}{v^2} \\ \{ttte, TTTTTT, eeee\} & \quad \text{if} \quad \begin{cases} \exists! A, \quad \frac{\Lambda_A}{\Delta_A} > \frac{1-v^2}{v^2} \\ \forall B \neq A, \quad \frac{\Lambda_B}{\Delta_B} < \frac{1-v^2}{v^2} \end{cases} \\ \{ttee, TTTTTTE, eeee\} & \quad \text{if} \quad \begin{cases} \text{for } I = A, B, \quad \frac{\Lambda_I}{\Delta_I} < \frac{1-v^2}{v^2} \\ \forall C \neq A, B, \quad \frac{\Lambda_C}{\Delta_C} > \frac{1-v^2}{v^2} \\ \mu_{AB} < 2v^2 - 1 \end{cases} \end{aligned}$$

Finally, let us consider the (light) case $v = 1$. In this case it is clear that, unlike (29), we have $(dt^A)^2 = 0$ so that *the coframe of the relativistic emission coordinate systems with $v = 1$ is of causal type $\{llll\}$* . It can be then shown that the other causal orientations c_A and C_{AB} are recovered by making $v = 1$ in (35). From expressions (35), because Λ_A and Δ_A are both positive and the μ_{AB} are all smaller than 1, the second members of the expressions for c_A vanish and those for the C_{AB} take the value 1, the c_A and the C_{AB} cannot but be space-like, $c_A = e$, $C_{AB} = E$. This result, obtained for an inertial homogeneous medium and four static clocks, may be shown true also for arbitrary clocks in general space-times [10]. We have thus: *all the relativistic positioning systems with light signals define in their whole domains a sole causal class, of causal*

signature

$$\{eeee, EEEEE, llll\}$$

These relativistic positioning systems, of great interest for future space research and navigation, have been considered elsewhere [8, 9, 10].

For same reasons that in the Newtonian case, the coordinate lines of emission coordinates are here also hyperbolas. Nevertheless, their causal types differ: meanwhile in the Newtonian case every hyperbola is everywhere time-like up to at its base point, where it is space-like, in the relativistic case with $v < 1$ the corresponding space-like point becomes enlarged to a whole space-like domain, bounded by two light-like points, the rest of the branches being time-like. In the relativistic case $v = 1$ the hyperbolas are spacelike everywhere. Obviously, this is at the basis of the richness (the above mentioned 103 causal classes) of the relativistic positioning systems.

7. Comments around our results

That the causal structure of the relativistic spacetime allows to locally classify coordinates systems in 199 causal classes is known from some time ago [1]. Nevertheless, the corresponding situation for Newtonian space-time has remained unanswered. We have here solve it, showing that in Newtonian space-time the number of causal classes of coordinate systems reduces to only 4 (theorem 2).

Of these four classes, the standard one, of causal signature $\{teee, TTTEE, teee\}$, seems to be the only class of which many people is aware or, at least, the only one having a physical interest.

We do not think so. On the contrary, notwithstanding its undeniable importance, we believe that their almost exclusive use in physics, reinforcing overly the space-time cut into space plus time, exaggerates the physical interest of the evolution vision (i.e. of the leading role of time dependence of spatial configurations in the description of space-time changes of physical systems).

Other cuts of the space-time may present, and presents, their intrinsic interest. It is the case, for example, of the Solar system synchronization, which foliates the space-time by time-like instants, as we have shown in subsection 5.2. And more importantly, also the case of the positioning systems, cutting any (history of an) extended object by four (histories of) electromagnetic pulses.

The very concept of synchronization, foliating space-time by instants not necessarily related to simultaneity, is revealed to be a gentle but powerful instrument which allows us to get in training to ‘see’ space-time under different, unconventional, viewpoints. In fact, as we have shown in subsection 5.1, the simple linear synchronization group is able to already generate coordinate systems of *any* Newtonian class.

Once became used to handle arbitrary synchronizations, one can try to learn to describe nature without using *any* synchronization at all. This is possible by means of the positioning systems. Although, of course, they can be *related* to standard coordinate systems, they do not *contain* their causal specific features, as reveals the fact that they

necessarily belong to *any* of the other three classes, the standard one being excluded, as we have seen in subsection 5.3.

But the role of Newtonian spacetime in modern gravitational physics is principally that of facilitate the comprehension of the analog features of the relativistic space-time. Consequently, in section 6, we have also considered the relativistic analogues of the above mentioned systems. 1) For simplicity we have applied the local Solar time synchronization to the relativistic rigidly rotating disk, and we have seen in subsection 6.2 that, in the interior of the light cylinder, the resulting causal class has same causal signature that the Newtonian one. 2) Concerning the linear synchronization group, we have shown in subsection 6.1 that it generates 29 causal classes, four of them having the same causal signature that the four Newtonian ones. 3) And finally, in contrast to the Newtonian case, we have seen in subsection 6.3 that positioning systems in relativity may be of 103 causal classes, three of them having the same causal signature that the corresponding 3 Newtonian causal classes, and only one of them, the {e e e e, E E E E E, l l l l} corresponding to relativistic positioning systems based in light signals.

The ability to take hold of Newtonian space-time without the use of the simultaneity foliation may seem rather academic. But such ability for the relativistic space-time seems urgent. Simply because, in relativity, relative simultaneity foliations, be them introduced as an approximate concept or as an exact one, have neither more nor less physical reality than the celestial crystal spheres of the Ptolemaic epicyclic theory of planets.

Such foliations are conventional constructions whose realization really demand the a priori knowledge of (a good number of) the physical quantities that usually one wants to know. As such constructions, they can play a role for the ‘a posteriori’ physical interpretation of some physical quantities, but are unusable as *starting* basis for referring physical observations of a unknown environment.

The direct confrontation of the physicists with their environment in order to know it gravitationally is a basic problem still unsolved in relativity. Such a confrontation needs a locating structure that, in order to not to chase its tail, be able to be constructed *before* the measure of the gravitational properties be done. As has been analyzed elsewhere (see, for example, [8, 9, 10]) this locating structure is constituted by the relativistic positioning systems broadcasting light signals in vacuum.

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